These notes are designed to give a fuller explanation of some of the ideas from the presentation slides. We will briefly discuss the orientibility of manifolds, show that the notion of smooth homotopy and isotopy is an equivalence relation, state some results of Brouwer related to fixed points of smooth maps, and mention the notion of degree of a map modulo 2.

1 Oriented manifolds

An orientation of a finite dimensional real vector space is an equivalence class of ordered basis such that the ordered basis \((b_1, \ldots, b_n)\) determines the same orientation as the basis \((b'_1, \ldots, b'_n)\) if the “change of basis” transformation from the former to the latter has positive determinant. Otherwise, when the determinant is negative, it determines the opposite orientation. The standard orientation of the vector space \(\mathbb{R}^n\) is determined by the ordered basis \((1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots, (0, \ldots, 0, 1)\). For zero dimensional vector spaces it is convenient to define an orientation as the symbol \(+1\) or \(-1\).

An oriented smooth manifold of dimension \(d\) is a \(d\)-dimensional manifold \(M\) together with a choice of orientation for each tangent space \(TM_x\). If \(d \geq 1\), they are required to fit together as follows: for each point of \(M\) there should exist a neighborhood \(U\) and a diffeomorphism \(h\) mapping \(U\) onto an open subset of \(\mathbb{R}^d\) (or \(\mathbb{H}^d\) if \(M\) has boundary) that is orientation preserving, in the sense that for each \(x \in U\) the isomorphism \(dh_x\) carries the orientation of \(TM_x\) to the standard orientation of \(\mathbb{R}^d\). A connected manifold has only two possible orientations. At the same time, not all manifolds are orientible. For example, the Mobius band is a 2-dimensional non-orientible manifold.

If \(M\) has boundary, there are three kinds of vectors in \(TM_x\) for a boundary point \(x\). These consists of the tangent vectors to the boundary: \(T(\partial M)_x\), the ‘outward’ normal vectors forming an open half space bounded by \(T(\partial M)_x\) and the ‘inward’ normal vectors forming the complementary half space. An orientation of \(M\) determines an orientation of \(\partial M\) as follows: for \(x \in \partial M\) choose a positively oriented basis \((x_1, \ldots, x_d)\) of \(TM_x\) such that \(x_1\) is an outward normal vector and \((x_2, \ldots, x_d)\) forms a basis of \(T(\partial M)_x\) (assuming \(d \geq 2\)). Then \((x_2, \ldots, x_d)\) determines an orientation of \(\partial M\). If the dimension of \(M\) is 1 then each boundary point \(x\) is assigned the orientation \(+1\) or \(-1\) according as a positively oriented vector at \(x\) points inward or outward.
Any complex manifold is orientable. This follows because if $T : \mathbb{C}^d \to \mathbb{C}^d$ is a $\mathbb{C}$-linear map, then under the isomorphism $(x_1 + iy_1, \ldots, x_d + iy_d) \leftrightarrow (x_1, \ldots, x_d, y_1, \ldots, y_d)$, $\mathbb{C}^d$ can be considered as the real vector space $\mathbb{R}^{2d}$. This induces a $\mathbb{R}$-linear map $T_{\mathbb{R}} : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$. $T_{\mathbb{R}}$ is always orientation preserving as the following lemma shows.

**Lemma 1.1.** If $T_{\mathbb{R}} : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ is as above then $\det(T_{\mathbb{R}}) = |\det(T)|^2$ where $\det(T)$ is the complex determinant of $T$.

**Proof.** Write $T = T_1 + iT_2$ where $T_1$ and $T_2$ are the real and imaginary parts of $T$. Then $T_{\mathbb{R}}$ has the block matrix form $\begin{pmatrix} T_1 & -T_2 \\ T_2 & T_1 \end{pmatrix}$. Now set $\overline{T} = T_1 - iT_2$ and define the matrices $P$ and $D$ as $P = \begin{pmatrix} I & iI \\ I & -iI \end{pmatrix}$; $D = \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}$. Here $I$ is the $n \times n$ identity matrix. A simple verification shows that $T_{\mathbb{R}} = P^{-1}DP$. After taking determinants on both sides, we get that $\det(T_{\mathbb{R}}) = \det(T)\det(\overline{T}) = \det(T)\det(T)$. \qed

## 2 Smooth homotopy and Isotopy

Recall that two smooth maps $f, g : M \to N$ are smoothly homotopic if there exists a smooth map $F : M \times [0,1] \to N$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$. We will show that this is an equivalence relation between smooth maps. The only non-trivial verification is that of transitivity. To this end, we require a smooth “bump” function $\lambda : \mathbb{R} \to [0,1]$ defined as follows:

$$\lambda(t) = \begin{cases} 
0 & t \leq 0 \\
\exp\{-\frac{1}{2}\} & t > 0
\end{cases}$$

Using $\lambda(t)$ we can define $\gamma(t) = \frac{\lambda(t - 1/3)}{\lambda(t - 1/3) + \lambda(2/3 - t)}$ for $0 \leq t \leq 1$. $\gamma(t)$ is smooth and satisfies $\gamma(t) = 0$ for $0 \leq t \leq 1/3$ while $\gamma(t) = 1$ for $2/3 \leq t \leq 1$. Given a homotopy $F : M \times [0,1] \to N$ between $f$ and $g$, and a homotopy $G : N \times [0,1] \to X$ between $g$ and $h$, we define

$$H(x,t) = \begin{cases} 
F(x,\gamma(2t)) & 0 \leq t \leq 1/2 \\
G(x,\gamma(2t - 1)) & 1/2 \leq t \leq 1
\end{cases}$$

$H$ is a smooth homotopy between $f$ and $h$. The same construction shows that the relation of smooth isotopy is an equivalence relation.

## 3 Further results involving degree

It was stated in the presentation that if $f : M \to N$ and $g : N \to X$ are smooth maps between equi-dimensional manifolds, satisfying all required hypothesis for the degree to be defined, then $\deg(g \circ f) = \deg(g)\deg(f)$. We will now verify this claim. First, choose a common regular value $y$ of $g$ and $g \circ f$, and notice that $(g \circ f)^{-1}(y) = \cup_{x \in g^{-1}(y)} f^{-1}(x)$. It is clear that each $x \in g^{-1}(y)$
is a regular value of $f$. Therefore

$$
\sum_{x \in g^{-1}(y)} \sum_{a \in f^{-1}(x)} \text{sign}(d(g \circ f)_a) = \sum_{x \in g^{-1}(y)} \sum_{a \in f^{-1}(x)} \text{sign}(dg_x) \text{sign}(df_a)
$$

$$
= \sum_{x \in g^{-1}(y)} \text{sign}(dg_x) \sum_{a \in f^{-1}(x)} \text{sign}(df_a)
$$

$$
= \sum_{x \in g^{-1}(y)} \text{sign}(dg_x) \text{deg}(f, x) = \text{deg}(f) \text{deg}(g)
$$

Now, suppose that $f, g : S^n \rightarrow S^n$ are smooth maps such that $f(x) \neq -g(x)$ for any $x \in S^n$. Then $f$ and $g$ are homotopic. The homotopy is given by

$$
F(x, t) = (1-t)f(x) + tg(x)
$$

This is well defined since $||(1-t)f(x) + tg(x)|| = 0$ implies that $f(x) = -\frac{1}{1-t}g(x)$. After taking norms we deduce that $t = \frac{1}{2}$ and hence $f(x) = -g(x)$, which is not possible. Consequently $\text{deg}(f) = \text{deg}(g)$. This little result has some neat consequences.

**Theorem 3.1 (Brouwer).** If $f : S^n \rightarrow S^n$ is a smooth map with degree different from $(-1)^{n+1}$ then $f$ has a fixed point.

**Proof.** Recall that the antipodal map $a(x) = -x$ has degree $(-1)^{n+1}$. If $f(x) \neq x$ for any $x \in S^n$ then $\text{deg}(f) = \text{deg}(a(x))$ by the little result above. This is a contradiction. \qed

Finally, in the spirit of the smooth hairy ball theorem, we will show further differences between odd and even dimensional spheres. Notice that the map

$$
v(x, x_1, \ldots, x_{2n-1}, x_{2n}) = (x_1, x, \ldots, x_{2n-1}, -x_{2n})
$$

from $S^{2n-1}$ into $S^{2n-1}$ has neither a fixed point nor maps any point $x$ to its antipode $-x$.\(^1\) This is not possible for even dimensional spheres.

**Theorem 3.2 (Brouwer).** If $f : S^{2n} \rightarrow S^{2n}$ is a smooth map then $f$ has a fixed point or maps some point to its antipode.

**Proof.** The identity map of $S^{2n}$ had degree 1 while the antipodal map has degree -1. If $\text{deg}(f) \neq 1, -1$ then $f$ must have both a fixed point and map some point to its antipode. Otherwise, if the degree of $f$ is 1 (resp. -1), then it maps some point to it’s antipode (resp. has a fixed point). \qed

### 4 Mod 2 degree of a map

The definition of degree that we have considered is called the Brouwer degree, and it requires that our manifolds be oriented. There is a related notion of degree, called the mod 2 degree of a map, that does not require the manifolds to be oriented. We assume that $M$ and $N$ are smooth

\(^1v\) is also a smooth non-vanishing tangent vector field on $S^{2n-1}$, which means that the hairy ball theorem is not true on odd dimensional spheres.
manifolds of equal dimension and that $M$ is compact, without boundary and $N$ is connected. For a smooth map $f : M \to N$, we consider the size of the fiber $f^{-1}(y)$ for any regular value $y$ and denote it by $\# f^{-1}(y)$. As we have seen, $\# f^{-1}(y)$ is locally constant. Furthermore, $\# f^{-1}(y)$ modulo 2 is independent of the choice of regular value and it is also homotopy invariant. The proof of these claims are very similar to the analogous claims for the Brouwer degree. As an example, suppose $F : M \times [0,1] \to N$ is a smooth homotopy between $f$ and $g$. We will show that for a common regular value $y$ of $f$ and $g$,

$$\# f^{-1}(y) \equiv \# g^{-1}(y) \quad (\text{mod } 2)$$

First assume that $y$ is a regular value of $F$. We note that $F^{-1}(y)$ is a compact 1-manifold with boundary $F^{-1}(y) \cap \partial(M \times [0,1]) = (0 \times f^{-1}(y)) \cup (1 \times g^{-1}(y))$. Since the boundary of compact 1-manifolds come in pairs as endpoints of arcs, it follows that

$$\# \partial(F^{-1}(y)) = \# f^{-1}(y) + \# g^{-1}(y) \equiv 0 \quad (\text{mod } 2)$$

In case $y$ is not a regular value of $F$, then there exists neighbourhoods $U_1$ and $U_2$ containing $y$ such that $\# f^{-1}(y)$ is constant of $U_1$ and $\# g^{-1}(y)$ is constant on $U_2$. By Sard’s theorem, $F$ has plenty of regular values in the open neighbourhood $U_1 \cap U_2$ of $y$. Choose any such regular value $y'$ of $F$ and apply the first case. We leave the claim that the mod 2 degree is independent of the choice of regular value to the reader. These two claims combine to provide a well-defined mod 2 degree of $f$, denoted $deg_2(f)$, as $\# f^{-1}(y) \pmod{2}$ for any regular value $y$.

5 Exercises

1. Let $f : S^n \to S^n$ be smooth and of odd degree. Prove that there exists a $x$ in $S^n$ such that $f(-x) = -f(x)$.

2. Accept the following (or prove it if you wish): If $f : S^n \to S^n$ is a smooth odd map i.e. $f(-x) = -f(x)$, then $deg(f)$ is odd. Using this, or otherwise, prove that if $f : S^n \to \mathbb{R}^n$ is smooth then $f(-x) = -f(x)$ for some $x$ in $S^n$.

3. Assuming the conclusion of problem 2, or otherwise, prove that given bounded measurable sets $A_1, \ldots, A_n$ in $\mathbb{R}^n$, there exists a hyperplane that bisects each of their areas.

4. Let $h : \Omega \to \mathbb{C}$ be a non-constant holomorphic map on a domain $\Omega \subset \mathbb{C}$. For $z_0 \in \Omega$, suppose that $h(z) - h(z_0)$ has a zero of order $k$ at $z_0$. Prove that $h$ is $k : 1$ in a deleted neighbourhood of $z_0$. Can this be generalized to non-constant holomorphic maps $h : \Omega \subset \mathbb{C}^n \to \mathbb{C}^n$? How does all this relate to multiplicity?

5. Consider the map $z \mapsto z^n$ from $S^1 \subset \mathbb{C}$ into $S^1$ where $n$ is an integer. This induces a homomorphism $\phi_n : \Pi_1(S^1) \to \Pi_1(S^1)$ of first homotopy groups. Compute this homomorphism and then compute the degree of the map $z \mapsto z^n$. What do you notice? Can you generalize this?

References