Morse Genericity and Morse’s Theorem for compact smooth manifolds.

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Supplementary: Storyline

Parametric-Transversality:
\[ F(x, s) \pitchfork Z \Rightarrow F_s \pitchfork Z \]

Morse Generic

\[ \exists \text{ not deg } V_T : \sum_{x \in V_T^{-1}(0)} \text{Ind}_p V_T = \chi(M) \]

\[ \sum_{p \in V^{-1}(0) \text{ sign}(DV(p)) \text{ homotopy invariant}} \text{ when } V \text{ at } p \in V^{-1}(0) \text{ is not degenerate} \]

Special case Poincare-Hopf:
\[ \chi(M) = \sum_{p \in V^{-1}(0)} \text{Ind}_p(V) \]

Morse Theorem for Morse f on M
\[ \chi(M) = \sum_{i=0}^{n} (-1)^i \text{cr}_i(f) \]

Non-degenerate zero \(\Leftrightarrow\) isolated
Statements of Main Theorems and Lemmas:

1. Constructive proof: Morse functions are dense in $C^\infty(M)$;

2. Parametric Transversality: If $F \pitchfork Z \subset N$ for $F : M \times S \to N$ and $F_s(x) := F(x, s)$, then $F_s \pitchfork Z$ for a.e. $s$, shortly almost every $s$;

3. Construction of vector field $V_T$ with $\sum_{p \in V_T^{-1}(0)} \text{Ind}_p(V_T) = \chi(M)$;

4. $\text{deg}(\partial U_p \ni x \mapsto \frac{V(x)}{|V(x)|}) = \text{sign}(\text{det}[DV](p))$ at n-d $p \in V^{-1}(0)$;

5. Poincare-Hopf via a homotopy between $V_T$ and a v.f $V_1$, e.g. $\nabla f$. 
Regular/Critical points, Transversality, Index, Morse funct

Here \( F \in C^\infty(M, N) \), \( Z \hookrightarrow N \), map \( F \) is **transversal** to \( Z \), shortly \( F \pitchfork Z \), when \( (DF)(p)T_pM + T_{F(p)}Z = T_{F(p)}N \) for \( p \in F^{-1}(Z) \) and if \( F(M) \supset Z = \{y\} \) then \( y \) is called **regular**. Our \( f \in C^\infty(M, \mathbb{R}) \), **critical** points \( \text{Cr}(f) := \{p : df_p = 0\} \) are **non-degenerate**, shortly n-d, when \( \det(\text{Hess}_p(f)) \neq 0 \) and \( \text{Ind}_p(f) := \# \) of negative eigenvalues of \( \text{Hess}_p(f) \); \( 0^*_M := 0\)-section of \( T^*M \ni (p, df(p)) =: j^1f(p) \). Finally, \( x = (x_1, \ldots, x_m) \) are local coordinates on \( M \) and \( f \) is **Morse** when all \( p \in \text{Cr}(f) \) are n-d's.
Examples of Morse functions: \( f(x) = x_1^2 - x_2^2 \) and \( f : S^{n-1} \ni x \to x_n \).

Claim: \( f \) is Morse iff \( j^1 f \pitchfork 0^*_M \).

Proof: Say \( \psi : M \times \mathbb{R}^m \to \mathbb{R}^m \) is the natural projection. Locally \( 0^*_M = M \times \{0\} \hookrightarrow M \times \mathbb{R}^m \). Therefore \( j^1 f \pitchfork 0^*_M \iff \) for \( p \in (j^1 f)^{-1}(0^*_M) \) linear maps \( \psi D(j^1 f)_p = \text{Hess}_p f : T_p M \to \mathbb{R}^m \) are surjective. 

\[ j^1f \text{ transverse to } 0^*_M \]
Morse property is generic: \( \forall \{f_j\}_{0 \leq j \leq N} \subset C^\infty(M, \mathbb{R}) \), all \( T^*_p(M) = \text{Span}_{\mathbb{R}}\{df_j(p)\}_{1 \leq j} \Rightarrow f_s(p) := f_0(p) + \sum_{1 \leq j} s_j f_j(p) \) is Morse for a.e. \( s \)

**Proof:** Via PTL with \( F(p, s) := (p, df_s(p)) \) since \( F \pitchfork O^*_M \) because maps

\[
\psi DF_{(x,s)}(\xi, \eta) = \sum_{1 \leq j} \eta_j df_j(x) + \text{Hess}_x f_s(\xi) \in \mathbb{R}^m \text{ are onto}. \]

**PTL = Parametric Transversality Lemma:** If \( F \in C^\infty(M \times S, N) \) and \( F \pitchfork Z \subset N \), then \( F_s \pitchfork Z \), where \( F_s(x) := F(x, s) \), for almost every \( s \).

**Main Theorem, Morse:** for \( f \) Morse \( \chi(M) = \sum_{i=0}^n (-1)^i cr_i(f) \), where \( cr_i(f) := \# \{ p \in Cr(f) : \text{Ind}_p(f) = i \} \) and \( m = \text{dim } M \).
Proof of PTL: Let \( W := F^{-1}(Z) \) and \( \pi \) be the restriction to \( W \) of projection \( M \times S \to S \). Using Sard’s Thm suffices to show that if \( b \) is a regular value for \( \pi \Rightarrow F_b \cap Z \). So, \( DF_{(x,b)}(T_{(x,b)}(M \times S)) + T_z Z = T_z N \)

for \( z := F_b(x) \in Z \) and we assume \( D\pi_{x,b} : T_{(x,b)} W \to T_b S \) is onto. Then

\[
\forall \ v_N \in T_z N \ \exists \ (v_M, v_S) \in T_{(x,b)}(M \times S) \text{ and } w \in T_x M \text{ s.th. both } v_N - DF_{(x,b)}(v_M, v_S), \ DF_{(x,b)}(w, v_S) \in T_z Z \text{ with } (w, v_S) \in T_{(x,b)} W .
\]

Then \( v_N - D(F_b)_x(v_M - w) = v_N - DF_{(x,b)}[(v_M, v_S) - (w, v_S)] \in T_z Z \)

Corollary: Restriction of generic height functions to \( M \hookrightarrow \mathbb{R}^n \) are Morse.
Application. Finding $\chi$ visually for surfaces $M_g \to \mathbb{R}^3$ of genus $g$:

For Morse function $f(x, y, z) = z$

$\text{Ind}_{\text{maxima}} = 2$, $\text{Ind}_{\text{minima}} = 0$

$\text{Ind}_{\text{saddle}} = 1 \Rightarrow \chi(M_g) = (-1)^0 + (-1)^1 \cdot 2g + (-1)^2 \cdot 1 = 2 - 2 \cdot g$

For a vector field $V$ on contractible nbd $U$ with

$U \cap V^{-1}(0) = \{p\}$ let

$\text{Ind}_p(V) := \text{deg}(\partial U \ni \frac{\phi_{V, p}}{||V(x)||} \in S^{m-1})$. When $\text{det}(DV(p)) \neq 0 \Rightarrow$
ϕ_{V,p} \text{ is homotopic to } x \mapsto \frac{DV(p)x}{|DV(p)x|} \Rightarrow \text{deg}(ϕ_{V,p}) = \text{sign}(\det[DV](p)) \, .

**Lemma:** Non-degenerate \( p \in Cr(f) \) are isolated.

**Indeed,** \( \nabla f(p) = 0 \) and \( \det(\text{Hess}_p(f)) \neq 0 \implies \exists \text{ nbhd } U \text{ of } p \) s.th. \( \nabla f|_U \) is a diffeo \( \Rightarrow \nabla f(x) \neq \nabla f(p) = 0 \forall x \in U \setminus \{p\} \) . ■

**Morse Theorem follows from Poincare-Hopf degree Thm:**

\[ \# V^{-1}(0) < \infty \text{ for vec. fields } V \text{ on } M \Rightarrow \chi(M) = \sum_{p \in V^{-1}(0)} \text{Ind}_p(V) \]

**Proof:** \( \text{sign}(\det[D(\nabla f)](p)) = (-1)^{\text{Ind}_p(f)} \) for \( f \) Morse, \( p \in Cr(f) \)

\[ P-H \text{ deg Thm} \Rightarrow \chi(M) = \sum_{x \in (\nabla f)^{-1}(0)} \text{Ind}_x(\nabla f) = \sum_i (-1)^i \text{cr}_i(f) \] . ■
Exists $V_T$ s.th. $\text{Ind}(V_T) := \sum_{p \in V_T^{-1}(0)} \text{Ind}_p(V_T) = \chi(M)$

and $\det([D V_T](p)) \neq 0$ for $p \in V_T^{-1}(0)$. **Construction of vec. f. $V_T$:**

We start with a triangulation $T(M)$ of $M$ (as in Vitali’s talk, page 8).

Denote $c_\sigma$ the centers and $U_\sigma$ nbhds of simplexes $\sigma$ with coordinates $(u, v)$ centered at $c_\sigma$ s.th. $c_\sigma \notin U_\sigma$ for simplexes $\tau$, $\dim \tau \geq \dim \sigma$ and $\tau \neq \sigma$; $\{v = 0\} \supset \sigma$, while $\dim \{v = 0\} = \dim \sigma$. On $U_\sigma$ we define vector fields $V_\sigma := \nabla(|u|^2 - |v|^2)$. We then inductively construct $V_k$ on ‘small’ nbds $U_k$ of the $k$-skeletons of $T(M)$, $k \geq 1$, by extending $V_{k-1}$
from nbhds $\mathcal{U}_{k-1}$ (perhaps shrinking the latter), and set $V_T := V_m$, 

$m = dimM$, namely: using the PofU we construct nonnegative $C^\infty$ functions $\psi_{k-1}$ and $\phi_k$ with supports in $\mathcal{U}_{k-1}$ and $\mathcal{U}_k$ s.th $\psi_{k-1} \equiv 1$

and $\psi_{k-1} + \phi_k \equiv 1$ on nbds of the $(k-1)$ and of the $k$-skeletons of $T(M)$

and define $V_k := \psi_{k-1} \cdot V_{k-1} + \phi_k \sum_{\{\sigma: dim \ \sigma = k\}} V_\sigma \Rightarrow V_T^{-1}(0) = \bigcup_\sigma \{c_\sigma\}$

(at $p \in \sigma$ with $dim \ \sigma = k$, $\psi_{k-1}(p) \cdot \phi_k(p) \neq 0$ we use that $-V_k(p) \not\in \mathbb{R}_+ \cdot \{V_{k-1}(p)\}$) \Rightarrow Ind_{c_\sigma} V_T = det(DV_T(c_\sigma) = (-1)^{dim \ \sigma}$ for all $\sigma$ \Rightarrow
\[ \text{Ind}(V_T) = \sum_{0 \leq k \leq m} (-1)^k s_k =: \chi(M), \text{ where } s_k := \# \{ \sigma : \text{dim} \sigma = k \}, \text{ as claimed.} \]

(Pict. below illustrates our construction.)  ■  Let \( V_0 := V_T \).

Next, constructions and proofs:

1. homotopy of \( V_0 \) to \( V_1 \),

2. then \( \text{Ind}(V_0) = \text{Ind}(V_1) \).

Note. We’ll use Riemannian metric on \( M \), functions \( \{f_j\}_{j \geq 1} \) from

"Morse property is generic" (page 6) and \( \Sigma_s(p) := \sum_{j \geq 1} s_j (\nabla f_j)(p) \) in a
similar way: using PT Lemma we conclude for a.e. \( s \in \mathbb{R}^N \), \( \Phi_s := (1 - t)V_0(p) + tV_1(p) + t(1 - t) \cdot \Sigma_s(p) \) and map \( F_s(p, t) := (p, \Phi_s(p, t)) \) from \( M \times [0,1] \to TM \), that \( F_s \cap 0_M \), where \( 0_M \) is the 0-section of \( TM \).

Consequently: at \( (p, t) \in \Phi_s^{-1}(0) \) maps \( D\Phi_s(p, t) \) are onto \( \Rightarrow \Phi_s^{-1}(0) \) is a finite union of smooth arcs \( \gamma \) closed, or with ends in and tangents \( \xi_\gamma \) transversal to \( M \times \{t\} \), \( t = 0,1 \) \( \Rightarrow \xi_\gamma(p, t) \in \ker[D\Phi_s](p, t) \). Pick continuous \( T_{\perp\gamma_p,t} \nsubseteq \xi_\gamma(p, t) \), \( (p, t) \in \gamma \), equal \( T_p M \times \{0\} \) at \( t = 0,1 \) \( \Rightarrow \Lambda_{p,t} := D\Phi_s(p, t)_{T_{\perp\gamma_p,t}} \) are isomorphisms equal \( DV_t(p) \) for \( t = 0,1 \).
We next show 2 for vector field $V_1$ not degenerate at all $p \in V_1^{-1}(0)$.

Moving along arc $\gamma$ positive at $\gamma$'s end continuous frame $\mathcal{F}_{p,t}$ in $T_{\perp}^{\gamma}p,t$, $(p, t) \in \gamma$, results in the oppositely or similarly oriented frame at the other end of $\gamma$ depending on $\gamma$ returning to the same value of $t$, i.e. 0 or 1, or not. But orientation of continuous $\Psi_{p,t} := \Lambda_{p,t}(\mathcal{F}_{p,t})$ in $TM$ is preserved.

Hence, due to the index being the sign of det of the tangent map (p. 9) it follows that the indexes at the ends of the 'returning' arcs cancel, while at the ends of the 'other' arcs equal $\Rightarrow \text{Ind}(V_0) = \text{Ind}(V_1)$.
\[ \text{Ind}(V_0) = \text{Ind}(V_1) \] is as required. See Picture:
References

Lectures on Morse Homology by Augustin Banyaga and David Hurtubise

Poincare-Hopf Degree

proof: http://math.uchicago.edu/~amwright/PoincareHopf.pdf

Partition of Unity Theorem:

http://isites.harvard.edu/fs/docs/icb.topic134696.files/Partitions_of_Unity.pdf

Morse Genericity: http://www.math.toronto.edu/mgualt/Morse