The maximal grade is 30. Note that you do not need to solve all the problems to get the maximal grade. **Duration – 3 hours**

1. (a) (3 points) What is the definition of Lebesgue outer measure of subsets of $\mathbb{R}$?
   
   (b) (3 points) Give a definition of measurable subsets of $\mathbb{R}$.
   
   (c) (3 points) Give a definition of measurable functions on $\mathbb{R}$.
   
   (d) (3 points) Give a definition of Lebesgue integral for functions defined on $\mathbb{R}$.

2. (a) (3 points) State Lebesgue dominated convergence theorem.
   
   (b) (3 points) State Fatou’s lemma.
   
   (c) (3 points) Give an example of the situation when the inequality in the text of Fatou’s lemma is strict.

3. For each of the following assertions determine if it is true or false. No proof of your answer is required.
   
   (a) (2 points) There exists an uncountable set $A \subset [0,1]$ of measure zero.
   
   (b) (2 points) For each measurable set $A \subset \mathbb{R}$ there exists an open set $U$ such that $A \subset U$ and $m(U \setminus A) \leq 0.01$.
   
   (c) (2 points) If $f$ is a measurable function on $[0,1]$, then there exists a measurable set $D \in [0,1]$ such that $mD > 0.9$, and a continuous function $g : [0,1] \rightarrow \mathbb{R}$ such that $f(x) = g(x)$ for each $x \in D$.

4. (4 points) Let $f$ be an integrable real-valued function on $[0,1]$ such that for every measurable subset $E$ of $[0,1]$ $\int_E f(x) \, dx = 0$. Prove that $f = 0$ almost everywhere on $[0,1]$. (Hint: Analyze sets $\{x \in [0,1] : |f(x)| \geq \frac{1}{n}\}$.)

5. Let $n_1 < n_2 < n_3 < \ldots$ be an infinite strictly increasing sequence of positive integer numbers.

   (a) (4 points) Let $A$ denote the set of all $x \in [0,2\pi]$ such that the sequence $\{\cos(n_k x)\}_{k=1}^{\infty}$ converges to a limit as $k \rightarrow \infty$. Prove that $A$ is measurable.

   (b) (4 points) Prove that the set $A$ cannot coincide with $[0, 2\pi]$. (In other words, the sequence of functions $\cos(n_k x)$, $k = 1, 2, \ldots$ cannot converge pointwise on $[0,2\pi]$.)