The de Rham Theorem

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The de Rham complex

Review.

Let $M^n$ be a smooth, oriented, triangulated $n$-manifold.

\[
\cdots \rightarrow \Omega^{k-1}(M) \xrightarrow{d_{k-1}} \Omega^k(M) \xrightarrow{d_k} \Omega^{k+1}(M) \rightarrow \cdots \quad \text{de Rham cochain complex}
\]

\[
\downarrow \text{Int}^{k-1} \quad \bigcirc \quad \downarrow \text{Int}^k \quad \bigcirc \quad \downarrow \text{Int}^{k+1}
\]

\[
\cdots \rightarrow \Sigma^*_{k-1} \xrightarrow{\partial_{k-1}^*} \Sigma^*_k \xrightarrow{\partial_k^*} \Sigma^*_{k+1} \rightarrow \cdots \quad \text{simplicial cochain complex}
\]

\[\text{Int}^\bullet : \Omega^\bullet(M) \rightarrow \Sigma^*_\bullet \text{ is a morphism of cochain complexes.}\]

**Theorem 1 (Elementary forms)**

\[\text{Int}^\bullet \text{ admits a right inverse, i.e. } \exists \Phi^\bullet : \Sigma^*_\bullet \rightarrow \Omega^\bullet(M) \text{ morphism of cochain complexes such that}
\]

\[\text{Int}^k \circ \Phi^k = \text{id}_{\Sigma^*_k} \quad \forall k.\]
The de Rham cohomology

**Definition.**

\[
H^k(M) := \ker d_k / \im d_{k-1} \quad k^{th} \text{ de Rham cohomology group}
\]

\[
H^k(\Sigma) := \ker \partial^*_k / \im \partial^*_{k-1} \quad k^{th} \text{ cohomology group of } \Sigma.
\]

**Remark.** As a morphism of cochain complexes, \( \text{Int}^k : \Omega^k(M) \longrightarrow \Sigma^* \) induces a well-defined homomorphism

\[
[\text{Int}^k] : H^k(M) \longrightarrow H^k(\Sigma) \quad \forall k.
\]

**Remark.** \( \ker(\text{Int}^\bullet) \) is a subcomplex of the de Rham cochain complex.

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**Lemma 1**

The subcomplex \( \ker(\text{Int}^\bullet) \) is acyclic, i.e. \( H^k(\ker(\text{Int}^\bullet)) = 0 \) \( \forall k \).

**Claim.** Lemma 1 is equivalent to

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**Lemma 1**

Let \( \omega \in \Omega^k(M) \) be closed and \( A \in \Sigma^*_{k-1} \) such that \( \text{Int}^k \omega = \partial^*_{k-1} A \).

Then \( \exists \alpha \in \Omega^{k-1}(M) \) such that \( d_{k-1} \alpha = \omega \) and \( \text{Int}^{k-1} \alpha = A \).
Proof of Claim. Assume that Lemma 1 holds and that $d_k \omega = 0$, $\text{Int}^k \omega = \partial^*_{k-1} A$. Then

$$\text{Int}^k(d_{k-1}(\Phi^{k-1} A)) = \partial^*_{k-1}(\text{Int}^{k-1}(\Phi^{k-1} A)) = \partial^*_{k-1} A = \text{Int}^k \omega.$$ 

Setting

$$\beta := d_{k-1}(\Phi^{k-1} A) - \omega \in \ker(\text{Int}^k) \cap \ker d_k,$$

we obtain,

$$[\beta] \in H^k(\ker(\text{Int}^\bullet)).$$

Lemma 1 implies that $[\beta] = 0$ and therefore, $\exists \gamma \in \ker(\text{Int}^{k-1})$ with $d_{k-1} \gamma = \beta$. Hence,

$$\omega = d_{k-1}(\Phi^{k-1} A - \gamma)$$

and, by setting $\alpha = \Phi^{k-1} A - \gamma$, we obtain

$$\text{Int}^{k-1} \alpha = \text{Int}^{k-1}(\Phi^{k-1} A) - \text{Int}^{k-1} \gamma = A.$$ 

Assume now that Lemma 1* is true and let $[\omega] \in H^k(\ker(\text{Int}^\bullet))$, i.e. $\omega \in \ker(\text{Int}^k) \cap \ker d_k$. Then by Lemma 1*, $\exists \alpha$ such that $d_{k-1} \alpha = \omega$ and $\alpha \in \ker(\text{Int}^{k-1})$. It follows that $[\omega] = 0$. \qed
The de Rham Theorem

**Theorem 2 (de Rham)**

\[\text{[Int}^k] : H^k(M) \longrightarrow H^k(\Sigma) \text{ is an isomorphism } \forall k.\]

**Proof.**

i) **[Int]^k** is surjective:
Let \([A] \in H^k(\Sigma)\). Set \(\omega := \Phi^k A \in \Omega^k(M)\). Since \(d_k \omega = \Phi^{k+1} \partial^* A = 0\), \([\omega] \in H^k(M)\).
Also, \([\text{Int}^k]\omega = [\text{Int}^k A] = \Phi^k A = A\).

ii) **[Int]^k** is injective:
Let \([\omega] \in \ker([\text{Int}^k])\). Then \(d_k \omega = 0\) and \([\text{Int}^k \omega] = [\text{Int}^k][\omega] = 0\), i.e. \(\text{Int}^k \omega \in \text{im } \partial^*_{k-1}\).
Lemma 1* implies that \(\omega\) is exact and thus, \([\omega] = 0\).

Thus, for completing the proof of de Rham’s theorem, it remains to show that \(\ker(\text{Int}^\bullet)\) is acyclic.

For that, we need the following two lemmas:
Lemma 2 (Closed forms in star-shaped sets)

Let $S$ be an open and *star-shaped* set in $\mathbb{R}^n$ and let $\omega$ be a closed $k$-form in $S$, $k > 0$. Then $\omega$ is exact.

*Proof.* Follows immediately from *Poincaré’s Lemma*, which we proved last time.

Lemma 3 (Extension of forms)

(a) Let $\omega$ be a closed $k$-form near $\partial \sigma$, where $\sigma = \sigma^s$ is an $s$-simplex in $\mathbb{R}^n$, $k \geq 0$, $s \geq 1$. Suppose that

$$\int_{\partial \sigma} \omega = 0 \quad \text{if } s = k + 1. \quad (1)$$

Then there is a closed $k$-form $\tilde{\omega}$ near $\sigma$ which extends $\omega$.

(b) Let $\omega$ be a closed $k$-form near the $s$-simplex $\sigma = \sigma^s \subset \mathbb{R}^n$, $k \geq 1$, $s \geq 1$, and let $\alpha$ be a $(k - 1)$-form near $\partial \sigma$ such that $d\alpha = \omega$ near $\partial \sigma$. Suppose that

$$\int_{\partial \sigma} \alpha = \int_{\sigma} \omega \quad \text{if } s = k. \quad (2)$$

Then there is a $(k - 1)$-form $\tilde{\alpha}$ near $\sigma$ such that $\tilde{\alpha}$ extends $\alpha$ and $d\tilde{\alpha} = \omega$ near $\sigma$. 
Proof by induction on $k$. We will show

i) $(a_0)$ holds

ii) $(a_{k-1}) \Rightarrow (b_k)$

iii) $(b_k) \Rightarrow (a_k)$, $k > 0$

i) Being a closed 0-form, $\omega$ is constant near any connected part of $\partial \sigma$.
   If $s > 1$, then $\partial \sigma^s$ is connected and $\omega$ equals a constant $c$ near $\partial \sigma$. Set $\tilde{\omega} = c$ near $\sigma$.
   If $s = 1$, and say $\sigma^1 = p_0 p_1$, then
   \[\omega(p_1) - \omega(p_0) = \int_{\partial \sigma} \omega = 0,\]
   by (1), and thus, $\omega$ equals a constant $c$ near $\partial \sigma$. Set $\tilde{\omega} = c$ near $\sigma$.

ii) Assume $(a_{k-1})$ holds and let $\omega, \alpha$ be as in $(b_k)$.
   By choosing a star-shaped neighborhood of $\sigma$ and applying Lemma 2, there exists a $(k-1)$-form $\alpha'$ near $\sigma$ such that $d\alpha' = \omega$ near $\sigma$.
   Set $\beta = \alpha - \alpha'$ near $\partial \sigma$ and observe that $d\beta = \omega - \omega = 0$. 
Notice that, if $s = k$, (2) and Stokes’ Theorem imply

$$\int_{\partial \sigma} \beta = \int_{\partial \sigma} \alpha - \int_{\partial \sigma} \alpha' = \int_{\sigma} \omega - \int_{\sigma} d\alpha' = 0.$$ 

By applying $(a_{k-1})$, we can extend $\beta$ to $\tilde{\beta}$, which is defined near $\sigma$ and closed. Setting $\tilde{\alpha} = \alpha' + \tilde{\beta}$ near $\sigma$, we obtain that $\tilde{\alpha}$ extends $\alpha$ and $d\tilde{\alpha} = \omega$ near $\sigma$ as we wished.

**iii)** Assume $(b_k)$, $k > 0$, holds and let $\omega$ be as in $(a_k)$.

Say $\sigma = p_0 \ldots p_s$ and set $\sigma' = p_1 \ldots p_s$. Let $\mathcal{P}$ be the union of all proper faces of $\sigma$ with $p_0$ as a vertex.

Choose now $\epsilon > 0$ small enough such that $\omega$ is defined in the $\epsilon$-neighborhood $U_\epsilon(\mathcal{P})$ of $\mathcal{P}$.

Since $U_\epsilon(\mathcal{P})$ is star-shaped, by Lemma 2, there exists a $(k - 1)$-form $\alpha'$ in $U_\epsilon(\mathcal{P})$ such that $d\alpha' = \omega$ in $U_\epsilon(\mathcal{P})$.

We have, in particular, $d\alpha' = \omega$ near $\partial \sigma'$.
If $s = k + 1$, setting $A = \partial \sigma - \sigma'$, we obtain $\partial A = -\partial \sigma'$ and

$$\int_{\sigma'} \omega - \int_{\partial \sigma'} \alpha' = \int_{\sigma'} \omega + \int_{\partial A} \alpha' = \int_{\sigma'} \omega + \int_{A} d\alpha' = \int_{\partial \sigma} \omega = 0$$

by (1). We can now apply (b$_r$) and extend $\alpha'$ to $\tilde{\alpha}'$ near $\sigma'$ such that $d\tilde{\alpha}' = \omega$ near $\sigma'$. It follows that there is a neighborhood $\tilde{U}$ of $\partial \sigma'$ in which $\alpha'$ and $\tilde{\alpha}'$ are defined and equal. Set

$$\alpha = \begin{cases} 
\alpha'|_{\tilde{U}} = \tilde{\alpha}'|_{\tilde{U}} & \text{in } \tilde{U} \\
\alpha' & \text{near } \mathcal{P}\setminus\tilde{U} \\
\tilde{\alpha}' & \text{near } \sigma'\setminus\tilde{U}
\end{cases}$$

Observe that $d\alpha = \omega$ near $\partial \sigma$. By means of a partition of unity extend $\alpha$ to $\tilde{\alpha}$ near $\sigma$. $\tilde{\omega} := d\tilde{\alpha}$ satisfies the required properties. \qed
Definition. Let $L^s$ denote the $s$-dimensional part of the triangulation of $M$, that is $L^s = \bigcup_i \sigma_i^s$.

Proof of Lemma 1*. We will define $\alpha_0, \ldots, \alpha_n$ such that

(a) $\alpha_s$ is defined near $L^s$, $s = 0, 1, \ldots, n$,
(b) $d\alpha_s = \omega$ near $L^s$, and $\alpha_s = \alpha_{s-1}$ near $L^{s-1}$, $s > 0$, and
(c) $\text{Int} \alpha_{k-1} = A$.

Then, $\alpha := \alpha_n$ is the required form.

Construct $\alpha_s$, $s = 0, 1, \ldots, n$, by induction on $s$:

By Lemma 2, there exists an $\alpha'_0$ near each vertex $q_i$ such that $d\alpha'_0 = \omega$.
If $k > 1$, set $\alpha_0 = \alpha'_0$.
If $k = 1$, for each vertex $q_i$ choose a number $b_i$ such that, setting $\alpha_0 = \alpha'_0 + b_i$ near $q_i$, $\text{Int} \alpha_0 = A$. 
Now suppose $\alpha_{s-1}$ has been constructed.

We will define $\alpha_s$ near each $s$-simplex such that (a), (b) and (c) hold. Since $\alpha_s$ is then fixed near $L^{s-1}$, we obtain a well-defined $\alpha_s$ near $L^s$.

Let $\sigma$ be an $s$-simplex. Then $d\alpha_{s-1} = \omega$ near $\partial \sigma$, by construction. If $s = k$, by (c),

$$\int_{\partial \sigma} \alpha_{k-1} = \operatorname{Int} \alpha_{k-1} \cdot \partial \sigma = A \cdot \partial \sigma = \partial^* A \cdot \sigma = \operatorname{Int} \omega \cdot \sigma = \int_\sigma \omega.$$

Since we can assume that $M$ is embedded in $\mathbb{R}^N$ for some $N \in \mathbb{N}$, we can now apply Lemma 3. It gives us a $(k - 1)$-form $\tilde{\alpha}_s$ near $\sigma$ such that $\tilde{\alpha}_s = \alpha_{s-1}$ near $\partial \sigma$ and $d\tilde{\alpha}_s = \omega$ near $\sigma$.

So (b) holds for $\tilde{\alpha}_s$. 

If $s \neq k - 1$, set $\alpha_s = \tilde{\alpha}_s$ near $\sigma$.
If $s = k - 1$, define $B = A - \text{Int} \tilde{\alpha}_{k-1}$ and set
\[
\alpha_{k-1} = \tilde{\alpha}_{k-1} + \Phi B \quad \text{near } L^{k-1}.
\]
To see that $\alpha_{k-1}$ satisfies (b), recall $\text{Supp}(\Phi \rho^*) \subset \text{St}(\rho)$ for each simplex $\rho$.
It follows that $\alpha_{k-1} = \tilde{\alpha}_{k-1}$ near $L^{k-2}$ and thus, $\alpha_{k-1} = \alpha_{k-2}$ near $L^{k-2}$.
Also,
\[
d\alpha_{k-1} = d\tilde{\alpha}_{k-1} + d\Phi B = d\tilde{\alpha}_{k-1} + \Phi \partial^* B = \omega \quad \text{near } L^{k-1}.
\]
Since
\[
\text{Int} \alpha_{k-1} = \text{Int} \tilde{\alpha}_{k-1} + B = A,
\]
(c) holds and $\alpha_s = \alpha_{k-1}$ is as we wished. \qed
An example: The Euler characteristic

**Definition.** Let $M^n$ be a manifold. The *Euler characteristic* $\chi$ of $M$ is the alternating sum

$$\chi(M) = \sum_{k=0}^{n} (-1)^k \text{dim}_\mathbb{R} H^k(\Omega).$$

The *de Rham Theorem* tells us that, no matter which triangulation we pick, the Euler characteristic equals the following:

$$\chi(M) = \sum_{k=0}^{n} (-1)^k \text{dim}_\mathbb{R} H^k(\Sigma),$$

where

$$0 \overset{}{\longrightarrow} \Sigma_0^* \overset{\partial_0^*}{\longrightarrow} \Sigma_1^* \overset{\partial_1^*}{\longrightarrow} \cdots \overset{\partial_{n-2}^*}{\longrightarrow} \Sigma_{n-1}^* \overset{\partial_{n-1}^*}{\longrightarrow} \Sigma_n^* \overset{}{\longrightarrow} 0$$

is the simplicial cochain complex according to the chosen triangulation of $M^n$. Using

$$\text{dim}_\mathbb{R} H^k(\Sigma) = \text{dim}_\mathbb{R} \ker \partial_k^* - \text{dim}_\mathbb{R} \text{im} \partial_{k-1}^*$$

and

$$\text{dim}_\mathbb{R} \Sigma_k^* = \text{dim}_\mathbb{R} \ker \partial_k^* + \text{dim}_\mathbb{R} \text{im} \partial_{k-1}^*,$$

we finally obtain

$$\chi(M) = \sum_{k=0}^{n} (-1)^k \text{dim}_\mathbb{R} \Sigma_k^*,$$

that is simply the alternating sum of the number of the $k$-dimensional faces, $k = 0, 1, \cdots, n$. 
Example 1. 2-Sphere

We can use a tetrahedron $T$ to triangulate $S^2$. Then

$$\chi(S^2) = \text{number of vertices of } T - \text{number of edges of } T + \text{number of faces of } T = 4 - 6 - 4 = 2.$$

Example 2. 2-Torus

Triangulate the torus in the following way:

$$\chi(T^2) = \text{number of vertices of } K - \text{number of edges of } K + \text{number of faces of } K = 9 - 27 + 18 = 0$$