MAT1124H, Random Matrix Theory, Bálint Virág

Lecture 1: Wigner's Semicircle Law

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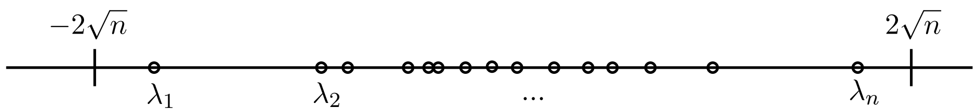
We want to consider **symmetric Wigner matrices**; more precisely, a symmetric Wigner matrix M has the following properties:

1. M is symmetric;
2. Each entry of M is a random variable;
3. $\{M_{ij}\}_{i \le j}$ are independent;
4. $\{M_{ij}\}_{i<j}$ have the same distribution; $\{M_{ii}\}_{i=1,\dots n}$ have the same distribution;
5. $E(M_{ij}) = 0$ for all i and j;
6. $E(M_{ij}^2) = 1$ for all $i \ne j$; $E(M_{ii}^2) = 2$ for all $i = 1, \dots n$;
7. M_{ij}'s moments are all finite, for all i and j.

Then we want to consider the eigenvalues of the matrix M. Let $\lambda_1 \le \lambda_2 \le \dots \le \lambda_n$ be the eigenvalues of M, then they form a point process on $[-2\sqrt{n}, 2\sqrt{n}]$. However, there is evidence that the distributions of the eigenvalues are not random; in particular, they are more concentrated around zero.



We will show that when n is big, the distribution of the eigenvalues approximates a semicircle distribution.

Before we make the theorem precise we need some notations. Let M be a symmetric Wigner matrix and A be a set on the real line. Let:

$$\ell^n(A) = E\left(\text{number of eigenvalues of } \frac{1}{\sqrt{n}} M_{n\times n} \text{ in } A\right)$$

Then ℓ^n is obviously a measure. Note that the normalization $\frac{1}{\sqrt{n}}$ pushes all the eigenvalues to the interval $[-2, 2]$. Also, we will define the semicircle distribution by the following density function:

$$f(x) = \frac{1}{2\pi}\sqrt{4 - x^2}; \quad x \in [-2, 2]$$

Note that we renormalized M by $\frac{1}{\sqrt{n}}$ because the semicircle distribution on $[-2, 2]$ has variance 1. We will use SC to denote the semicircle distribution from now on.

Theorem 1 (Wigner's Semicircle Law, Average Version)**.**

$$\frac{\ell^n(A)}{n} \overset{weak}{\Rightarrow} SC \text{ as } n \to \infty$$

The proof uses the methods of moments. Let:

$$L^n(A) = \text{number of eigenvalues of } \frac{1}{\sqrt{n}} M_{n\times n} \text{ in } A$$

Where A is a subset of the real line. Then L^n is a random measure and it equals to the sum of n point-masses, i.e.:

$$L^n = \sum_{i=1}^{n} \delta_{\lambda_i}$$

Furthermore we see that:

$$E\left(L^n(A)\right) = \ell^n(A)$$

By construction. We will also use the following notation:

$$\langle f, \mu \rangle = \int f(x) d\mu(x)$$

The idea of the proof is as follows. Let $k \in \mathbb{N}$ be fixed. Then:

$$\left\langle x^k, L^n \right\rangle = \sum_{i=1}^{n} \lambda_i^k = \mathrm{Tr}\left[\left(\frac{1}{\sqrt{n}} M_{n\times n}\right)^k\right]$$

We will show that:

$$\left\langle x^k, \ell^n \right\rangle = E\left(\left\langle x^k, L^n \right\rangle\right) = E\left(\mathrm{Tr}\left[\left(\frac{1}{\sqrt{n}} M_{n\times n}\right)^k\right]\right)$$

Then show that:

$$\frac{1}{n}\left\langle x^k, \ell^n \right\rangle \to \left\langle x^k, SC \right\rangle \text{ as } n \to \infty$$

Once we establish this for any $k \in \mathbb{N}$, then we will use the following lemma:

Lemma 2. *Let μ_n be a sequence of probability measures with bounded support; furthermore,*

$$\left\langle x^k, \mu_n \right\rangle \to \left\langle x^k, \mu \right\rangle \textit{ as } n \to \infty$$

For all $k \in \mathbb{N}$. Then

$$\mu_n \overset{weak}{\Rightarrow} \mu$$

Using this lemma we see that

$$\frac{1}{n}\ell^n \overset{\text{weak}}{\Rightarrow} SC$$

We will now begin the proof of Theorem 1

Proof. Consider:

$$\frac{1}{n}\left\langle x^k, \ell^n \right\rangle = \frac{1}{n} E\left(\mathrm{Tr}\left[\left(\frac{1}{\sqrt{n}} M_{n\times n}\right)^k\right]\right) = \frac{1}{n^{1+\frac{k}{2}}} \sum_{i_1, i_2, \ldots, i_k = 1}^{n} E\left(M_{i_1 i_2} M_{i_2 i_3} \ldots M_{i_k i_1}\right)$$

The key point is to analyze the sum on the right. Note that $E(M_{ik}) = 0$ for all i, j. Therefore, in order for the sum on the right to be nonzero, we need to repeat every term at least once; for example, if $M_{i_1 i_2}$ is different from all of $M_{i_2 i_3} \ldots M_{i_k i_1}$, then:

$$E\left(M_{i_1 i_2} M_{i_2 i_3} \ldots M_{i_k i_1}\right) = E\left(M_{i_1 i_2}\right) E\left(M_{i_2 i_3} \ldots M_{i_k i_1}\right) = 0$$

Now, consider:

$$\frac{1}{n^{1+\frac{k}{2}}}\sum_{i_1,i_2,\ldots,i_k=1}^{n} E\left(M_{i_1 i_2}M_{i_2 i_3}\ldots M_{i_k i_1}\right)$$
$$=\frac{1}{n^{1+\frac{k}{2}}}\sum_{\text{double path}} E\left(M_{i_1 i_2}M_{i_2 i_3}\ldots M_{i_k i_1}\right)+\frac{1}{n^{1+\frac{k}{2}}}\sum_{\text{other paths}}^{n} E\left(M_{i_1 i_2}M_{i_2 i_3}\ldots M_{i_k i_1}\right)$$

We will give a combinatorial argument. For paths that do not go over every vertex twice (i.e. the 'other paths'), they can meet at most $\left\lfloor\frac{k+1}{2}\right\rfloor$ vertices. Therefore,

$$\frac{1}{n^{1+\frac{k}{2}}}\sum_{\text{other paths}}^{n} E\left(M_{i_1 i_2}M_{i_2 i_3}\ldots M_{i_k i_1}\right)\leq\frac{1}{n^{1+\frac{k}{2}}}C_k\frac{n!}{\left\lfloor\frac{k+1}{2}\right\rfloor!\left(n-\left\lfloor\frac{k+1}{2}\right\rfloor\right)!}\leq\frac{1}{n^{1+\frac{k}{2}}}C_k' n^{\left\lfloor\frac{k+1}{2}\right\rfloor}\to 0$$

As $n\to 0$ for some $C_k, C_k'<\infty$. For the double path sum, note each path passes through exactly $k/2$ different vertices twice. There are $\binom{n}{k/2}$ ways to choose these vertices, and m_k ways to select the paths once the vertices are fixed, where $m_k=C_{k/2}$ if k is even and 0 if k is odd. Note that C_n is the Catalan number given by the formula:

$$C_n=\frac{1}{n+1}\left(\begin{array}{c}2n\\ n\end{array}\right)$$

(It is an exercise to show that the combinatorial statement about is true.) Therefore,

$$\frac{1}{n^{1+\frac{k}{2}}}\sum_{\text{double path}} E\left(M_{i_1 i_2}M_{i_2 i_3}\ldots M_{i_k i_1}\right)=\frac{1}{n^{1+\frac{k}{2}}}m_k n(n-1)(n-2)\ldots(n-(k/2+1)+1)$$
$$\sim\frac{1}{n^{1+\frac{k}{2}}}m_k n^{k/2+1}=m_k$$

In summary, we have shown that:

$$\lim_{n\to\infty}\frac{1}{n}\left\langle x^k,\ell^n\right\rangle=m_k$$

Now we want to show that

$$\left\langle x^k, SC\right\rangle=m_k$$

To do this, consider a simple random walk on $\mathbb{Z}$ starting at 0, and let:

$$T=\min\{t: X_t=-1\}$$

Now fix z with $|z|<1$. Consider:

$$E\left(z^{T-1}\right)=g(z)=\sum_{n=0}^{\infty}z^k P(T-1=k)=\sum_{n=0}^{\infty}\frac{1}{2^{k+1}}z^k m_k$$

If we can show that:

$$g(z)=\sum_{n=0}^{\infty}\frac{1}{2^{k+1}}z^k m_k=\sum_{k=0}^{\infty}\frac{1}{2^{k+1}}z^k S^k=\frac{1}{2}E\left(\frac{1}{1-\frac{zS}{2}}\right)=\frac{1}{2\pi}\int_{-2}^{2}\frac{1}{1-\frac{zs}{2}}\sqrt{4-s^2}\,ds$$

Where $S\sim SC$, then $\left\langle x^k,SC\right\rangle=m_k$ for all k and we complete the proof. To do this, first note that:

$$E\left(z^T\right)=zg(z)=\frac{1}{2}z+\frac{1}{2}zE\left(z^{T_1+T_2}\right)$$

Where T_1, T_2, T are independent and have the same distributions. Therefore we get:

$$zg(z)=\frac{1}{2}z\left(1+(z(g(z))^2\right)$$

Which is a quadratic equation. Solving yields:

$$g(z)=\frac{1\pm\sqrt{1-z^2}}{z^2}$$

However, by definition $g(0)=E(0^{T-1})=1$, so our only solution is:

$$g(z)=\frac{1-\sqrt{1-z^2}}{z^2}$$

All there is left to show (exercise) is that:

$$g(z)=\frac{1-\sqrt{1-z^2}}{z^2}=\frac{1}{2\pi}\int_{-2}^{2}\frac{1}{1-\frac{zs}{2}}\sqrt{4-s^2}\,ds$$

Which will complete the proof. □

Theorem 3 (Wigner's Semicircle Law)**.**

$$\frac{1}{n}L^n\Rightarrow SC\ \text{as}\ n\to\infty$$

Weakly in probability.

Proof. It is sufficient to show that:

$$\frac{1}{n}\left\langle x^k,L^n\right\rangle-\frac{1}{n}\left\langle x^k,\ell^n\right\rangle\xrightarrow{P}0$$

For every k. To do that, we will show that:

$$E\left[\left(\frac{1}{n}\left\langle x^k,L^n\right\rangle\right)^2\right]-\left(E\left[\frac{1}{n}\left\langle x^k,L^n\right\rangle\right]\right)^2\to 0\text{ as }n\to\infty$$

Then Chebyshev's Inequality yields the desired result. Note that:

$$E\left[\left(\frac{1}{n}\left\langle x^k, L^n\right\rangle\right)^2\right] - \left(E\left[\frac{1}{n}\left\langle x^k, L^n\right\rangle\right]\right)^2 = \frac{1}{n^{2+k}}\left(E\left[\left(\operatorname{Tr} M^k\right)^2\right] - \left(E\left[\operatorname{Tr} M^k\right]\right)^2\right)$$

$$= \frac{1}{n^{2+k}}\left(\begin{array}{l} E\left[\left(\sum_{i_1,i_2,\ldots,i_k=1}^{n}\left(M_{i_1i_2}M_{i_2i_3}\ldots M_{i_ki_1}\right)\right)\left(\sum_{j_1,j_2,\ldots,j_k=1}^{n}\left(M_{j_1j_2}M_{j_2j_3}\ldots M_{j_kj_1}\right)\right)\right] \\ -\left(\sum_{i_1,i_2,\ldots,i_k=1}^{n} E\left(M_{i_1i_2}M_{i_2i_3}\ldots M_{i_ki_1}\right)\right)\left(\sum_{j_1,j_2,\ldots,j_k=1}^{n} E\left(M_{j_1j_2}M_{j_2j_3}\ldots M_{j_kj_1}\right)\right)\end{array}\right)$$

$$= \frac{1}{n^{2+k}}\sum_{i\leq j}\operatorname{Cov}\left(\sum_{i_1,i_2,\ldots,i_k=1}^{n}\left(M_{i_1i_2}M_{i_2i_3}\ldots M_{i_ki_1}\right), \sum_{j_1,j_2,\ldots,j_k=1}^{n}\left(M_{j_1j_2}M_{j_2j_3}\ldots M_{j_kj_1}\right)\right)$$

$$= \frac{1}{n^{2+k}}C_k n^{k+1} \to 0 \text{ as } n \to \infty$$

Where the last step requires a combinatorial argument similar to our proof for Theorem 1. □