Preliminaries to Chow’s Theorem

Changho Han

MAT 477

January 16, 2014
Proof of Riemann Extension Thm

From Fact 1, $X = X^r \cup X'$ with \( \dim X^r = r \), $X'$ ∗-analytic of \( \dim < r \).

Take $a \in X^r \setminus X'$. WLOG, assume $a = 0$, $X^r = \mathbb{C}^r \subset \mathbb{C}^r \times \mathbb{C}^{n-r}$ locally.

$f$ analytic on $\{ \bar{x} \} \times (\mathbb{C} \setminus \{0\})$, so $f$ extends analytically to $\{ \bar{x} \} \times \mathbb{C}$ via:

$$f(\bar{x}, x_n) := \int_{|z|=1} f(\bar{x}, z) \frac{dz}{z - x_n}$$

As $\bar{x}$ varies, $f(\bar{x}, z)$ for fixed $z \neq 0$ varies analytically, so that $f$ extends analytically on $X^r \setminus X'$. Then, by induction on $\dim X$, $f$ extends analytically to $U$.\qed
Proof of Mumford’s Lemma

Pick an open nbhd $U$ of $f^{-1}(y)$ with $\overline{U}$ compact, and let $\partial U$ be its bdry.

Let $(V_n)$ be decreasing sequence of open nbhds of $y$ with $\overline{V_n}$ compact and $\cap V_n = \{y\}$. Then, $\cap_n(f^{-1}(\overline{V_n}) \cap \partial U) = f^{-1}(y) \cap \partial U = \emptyset$. But this is an intersection of countable # of cpcts, so $f^{-1}(\overline{V}) \cap \partial U = \emptyset$ for $V = V_m$.

Let $g : U \cap f^{-1}(V) \to V$ be restriction of $f$. Then, $\forall K \subset U$ compact, $g^{-1}(K) = U \cap f^{-1}(K)$ is compact because $f^{-1}(K)$ is closed, contained in $\overline{U}$ compact and $f^{-1}(K) \cap \partial U \subset f^{-1}(\overline{V}) \cap \partial U = \emptyset$. □
Properties of Resultant

Res is a matrix, and res is a determinant of entries in $S$, so that if $\exists$ a map $ev_a : S \ni f \mapsto f(a)$, then this commutes with both Res and res.

Claim: $\forall P, Q \in \mathbb{C}[w], \text{res}(P, Q) = 0 \iff P, Q$ have a common root.

Proof: Denote $p = \deg P$, $q = \deg Q$. $\text{res}(P, Q) = 0 \Rightarrow \exists F, G \in \mathbb{C}[w]$ nonzero, $\deg F < q$, $\deg G < p$ with $FP + GQ = 0$, so $FP = -GQ$. Since $\deg F < \deg Q$, not all roots of $Q$ are roots of $F$, so some roots of $Q$ are roots of $P$. Conversely, if $P, Q$ have a common root, then trivial. □
Suppose $c \in \mathbb{C}^d$, $w \in \mathbb{C}$ and $P(a, w) := w^d + \sum_{j=1}^{d} a_j w^{d-j}$. Then, for $F \in \mathbb{C}\{\bar{x}, w\}$, $\exists ! Q \in \mathbb{C}\{\bar{x}, a, w\}$ and $r_j \in \mathbb{C}\{\bar{x}, a\}$ for $1 \leq j \leq d$ so that $F(\bar{x}, w) = Q(\bar{x}, a, w)P(c, w) + \sum_{j=1}^{d} r_j(\bar{x}, a)w^{d-j}$. (*)

**Proof of Weierstrass Division Thm** using **Weierstrass Prep Thm**:

$f = uP$ with $u(0) \neq 0$ and $P \in \mathbb{C}\{\bar{x}\}[x_n]$. It suffices to divide $g$ by $P$ via **Special Weierstrass Division Thm**.
Proof of Weierstrass Preparation Thm

Take $F = f$, $w = x_n$. Using Special Weiers. Div. Thm look for solution of $r_j(x, a(x)) = 0$ with $a(0) = 0 \forall j$. Then, if $r_j(0, 0) = 0 \forall j$ and $\det(\frac{\partial r_j}{\partial a_k}(0, 0)) \neq 0$, then by Implicit Function Thm, we can find such solution. Indeed, set $x = 0$ and $a = 0$, then by comparing degrees in

$$x_n^d(\alpha + \cdots) = Q(0, 0, x_n)x_n^d + \sum r_j(0, 0)x_n^{d-j} \Rightarrow r_j(0, 0) = 0 \forall j.$$
Similarly, $Q(0, 0, 0) = \alpha \neq 0$. Taking $\frac{\partial}{\partial a_k}$ of (*) for each $k$ yields

$$0 = Q(0, 0, x_n)x_n^{d-k} + \frac{\partial Q}{\partial a_k}(0, 0, x_n)x_n^d + \sum_j \frac{\partial r_j}{\partial a_k}(0, 0)w^{d-j}.$$  For $j > k$

by comparing degrees $\Rightarrow \frac{\partial r_j}{\partial c_k}(0, 0) = 0$; for $j = k$, $\frac{\partial r_j}{\partial c_j}(0, 0) = \alpha \neq 0$.

Summarizing, $(\frac{\partial r_j}{\partial c_k}(0, 0))$ is an upper triangular matrix with nonzero diagonal entries. Hence, $\det(\frac{\partial r_j}{\partial c_k}(0, 0)) \neq 0$, and this proves the thm.  \qed